Fine-Tuning Ideal Worlds for the Xor of Two Permutation Outputs

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Outline

Luby-Rackoff Problem

- Feistel and Coppersmith: designed IBM's Lucifer cipher using Feistel networks
- Luby and Rackoff: analyzed Feistel network when the round function is a secure pseudorandom function (PRF)
  - 3 rounds: a pseudorandom permutation (PRP),
  - 4 rounds: a strong pseudorandom permutation
- Luby-Rackoff problem: how to make secure PRPs from secure PRFs?

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Luby-Rackoff Backward Problem

- AES is everywhere nowadays
  - AES, or any other block ciphers, is typically modeled as a PRP
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XoP1 and XoP2

- How to build secure PRFs from secure PRPs?

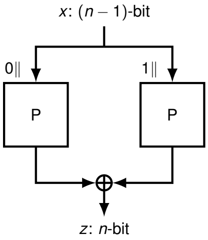

Figure 1: XoP1 based on a single (keyed) PRP: P

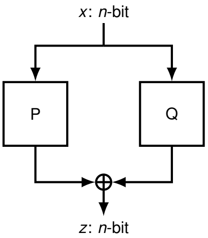

Figure 2: XoP2 based on two (keyed) PRPs: P and Q

Applications

- Symmetric-key primitive designs to achieve beyond birthday-bound (BBB) security
- MACs: nEHtM [DNT19], DbHtS [DDNP18], EWCDM [CS16]
- AEADs: CWC+ [DNT19], SCM [CLLL21], XOCB [BH+23]

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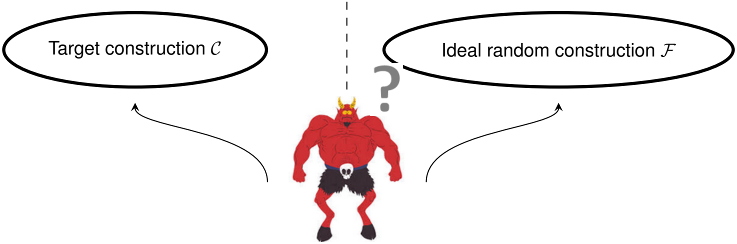

attacker $\mathcal{A}$

- $\mathcal{A}$ makes q queries to the construction oracle ($\mathcal{C}$ or $\mathcal{F}$)
- Security: a distinguishing probability of the two worlds:
  - $\mathbf{Adv}^{\mathrm{su}}_{\mathcal{C}}(\mathcal{A})$ can be denoted as a function of q
- $\mathbf{Adv}^{\mathrm{su}}_{\mathcal{C}}(\mathcal{A})$ is negligible $\Longrightarrow$ $\mathcal{C}$ is secure

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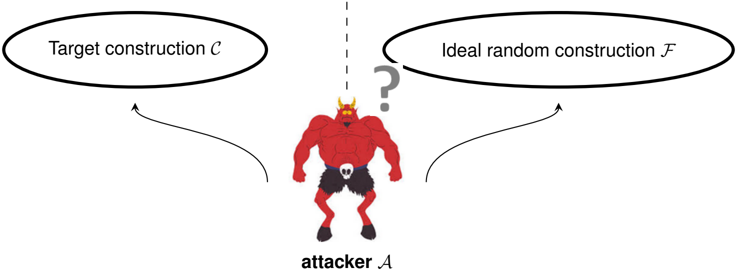

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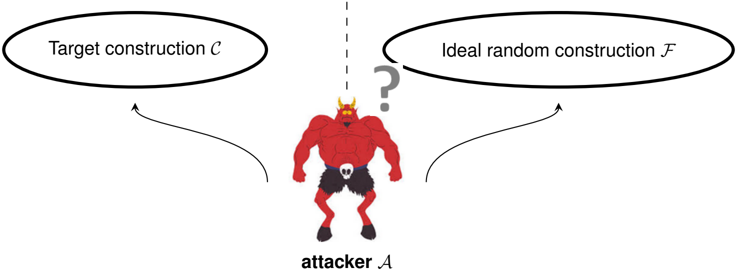

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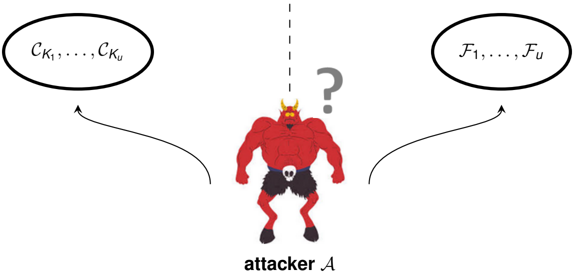

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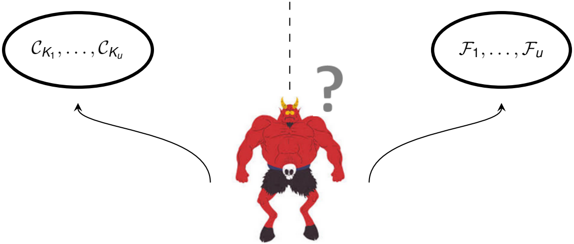

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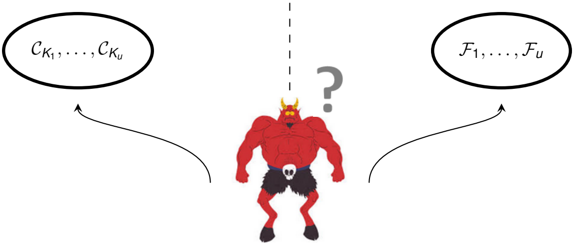

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PRF Security of XoP

- u: number of users
- q: total number of queries
- q_m: maximum number of queries per instance, $q_m \leq q \leq uq_m$
- P, Q: n-bit random permutations
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- Generates tag to authenticate a given message
- Protects data integrity by verifying tag value
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  - Deterministic: DbHtS [DDNP18, ToSC '18]
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DbHtS and nEHtM

- $\mathrm{H} = (\mathrm{H}^1, \mathrm{H}^2) : \{0,1\}^{2k} \times \mathcal{M} \to \{0,1\}^{n-1} \times \{0,1\}^{n-1}$:
  a $(2n-2)$-bit hash function where $\mathrm{H}_{K_h}(M) = (\mathrm{H}^1_{K_{h_1}}(M), \mathrm{H}^2_{K_{h_2}}(M))$
- $E : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$: a block cipher

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$$\mathrm{DbHtS}[\mathrm{H}_{K_h}, \mathrm{E}_K](M) \stackrel{\text{def}}{=} \mathrm{E}_K(0 \| \mathrm{H}^1_{K_{h,1}}(M)) \oplus \mathrm{E}_K(1 \| \mathrm{H}^2_{K_{h,2}}(M))$$

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- Unforgeability
  - Infeasible to generate a new valid message/tag pair
  - Allow q authentication queries and v verification queries to an adversary
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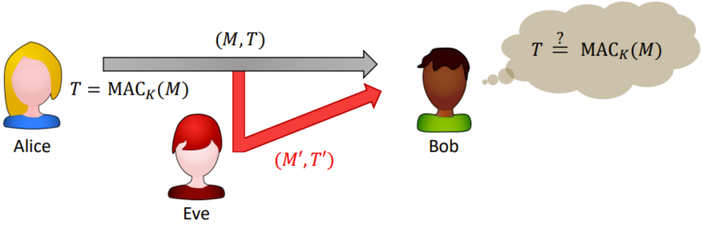
  - Infeasible to distinguish from a random variable-input-length (VIL) function up to $(q+v)$ queries
  - $\Rightarrow$ a secure MAC, i.e., unforgeable



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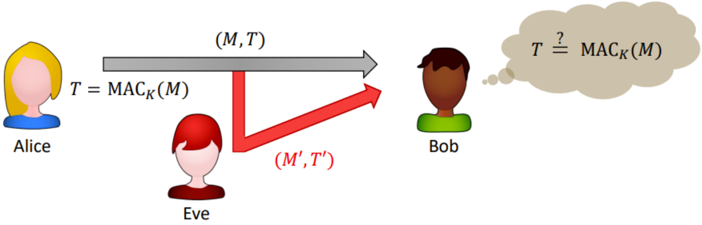

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MAC Security

- Both cases need to show PRF security for q authentication queries

- But they are XoP1-based! So the adversarial advantage always exceeds $q/2^n$

Do Independent Two Permutations Yield Better Security Bound?

- $q/2^n$ is n-bit security, i.e., fully secure in some sense
- However, this is not always true in the multi-user setting
- [CCL23, Crypto '23] shows XoP2-based nEHtM may have a better (PRF) security bound than XoP1-based nEHtM for $q = uq_m$ case
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- We conjecture XoP1 and XoP2 enjoy the (almost) same security bound by fine-tuning!

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Security Comparison

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Fine-Tuning Mirror Theory

- Mirror theory can give a sharp lower bound of the number of solutions, $h(\Gamma)$, to the given system Γ (and let $N = 2^n$):

$$h(\Gamma) \geq \frac{(N)_{q_P}}{N^q}$$

- In the "fine-tuned" ideal world, we want to have $\epsilon \ll \frac{q}{N}$ s.t.

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- From direct derivation, we have the $\frac{q}{N}$ term, which is what we want to avoid

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Proving MAC Security through PRF*

- PRF* security suffices for MAC security
  - authentication queries $\approx$ random function queries not outputting 0^n
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- (Multi-user) MAC security using the fine-tuned ideal world!
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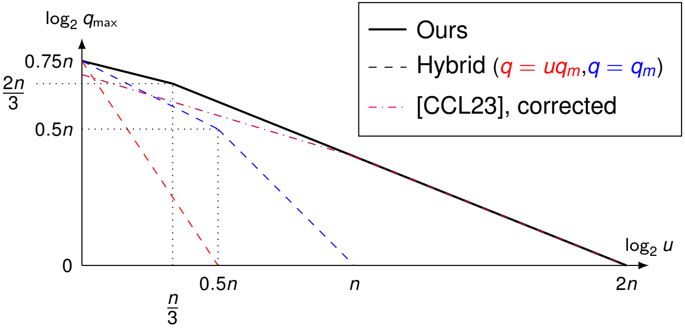
Improved mu-MAC Security of nEHtM

- Security of nEHtM, in terms of the thresholds

	#Query q_m	#User u	#Perm	Security
[DNT19]	$2^{0.66n}$	-	1	su-MAC
[CLLL20]	$2^{0.75n}$	-	1	su-MAC
[CCL23]	$2^{0.7n}$	2^{2n} (*)	2	mu-PRF
This work	$2^{0.75n}$	2^{2n}	1	mu-MAC

  - The query threshold q_m is per user (u is small for mu security)
  - The user threshold u is for small q_m
  - The bug (*) in [CCL23] is corrected in our paper

Graphical Comparison



Previous Multi-User PRF Security of DbHtS

- [SWGW21, Crypto '21] proved $2/3n$-bit security
- Their DbHtS construction assumes the underlying hash function is regular, AU, and based on ideal cipher

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- We prove mu-PRF* security of DbHtS in both settings: [SWGW21, Crypto '21] and [DDNT23, ToSC '23]
- ...but introducing q_m to achieve better bounds!
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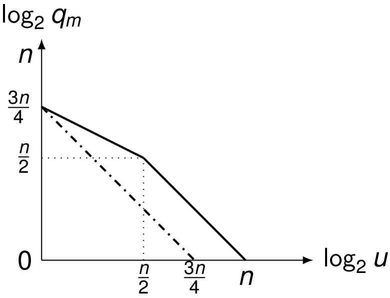
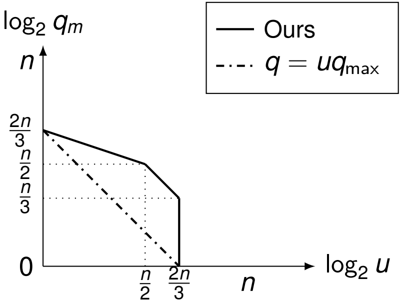
- The right figure compares our result with [DDNT23] and the left one compares ours with [SWGW21]





Conclusion

New results

- Deeper understanding about fundamental XoP1
- New tighter fine-tuned extended mirror theory with an upper bound for n-bit or any $\xi_{\max}$
- Improved multi-user security of nEHtM and DbHtS
- Fixing a flaw in the previous multi-user result of nEHtM

Future research

- Fine-tuning other security notions (Encryption?)
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Thank you for your attention!

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Chi-Squared Method [DHT17]

- $Z^i_{\mathcal{S}}$: a random variable over Ω that follows
  - the distribution of the i-th answer obtained by $\mathcal{A}$ interacting with $\mathcal{S}$

$$\mathsf{p}^{\mathbf{z}}_{\mathcal{S}}(z) \stackrel{\text{def}}{=} \Pr\left[Z^i_{\mathcal{S}} = z \mid (Z^1_{\mathcal{S}}, \ldots, Z^{i-1}_{\mathcal{S}}) = \mathbf{z}\right]$$

- Chi-squared method:

$$\|\mathsf{p}_{\mathcal{S}_0}(\cdot) - \mathsf{p}_{\mathcal{S}_1}(\cdot)\| \leq \left(\frac{1}{2}\sum_{i=1}^{q} \mathop{\mathbf{Ex}}_{\mathbf{z}}\left[\chi^2(\mathbf{z})\right]\right)^{\frac{1}{2}}$$

where the expectation is taken over the real world and

$$\chi^2(\mathbf{z}) \stackrel{\text{def}}{=} \sum_{z \in \Omega} \frac{\left(\mathsf{p}^{\mathbf{z}}_{\mathcal{S}_1}(z) - \mathsf{p}^{\mathbf{z}}_{\mathcal{S}_0}(z)\right)^2}{\mathsf{p}^{\mathbf{z}}_{\mathcal{S}_0}(z)}$$

Patarin's H-coefficient Technique

- For any good transcript z, it holds

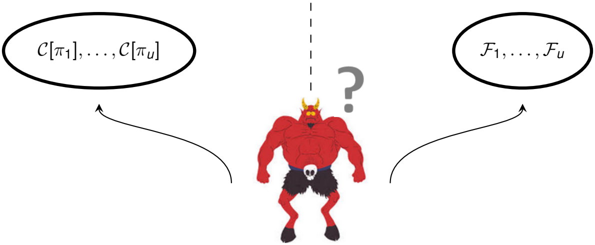
$$\frac{P_{S_1}(z)}{P_{S_0}(z)} \geq 1 - \epsilon$$

- Then we have

$$\mathbf{Adv}(\mathcal{A}) \leq \epsilon + \Pr[Z_{\mathcal{S}_0} \in \mathcal{T}_{\text{bad}}]$$

- $\mathcal{T}_{\text{bad}}$ and ϵ: depend on the construction
- $\Pr[Z_{\mathcal{S}_0} \in \mathcal{T}_{\text{bad}}]$: a combinatorial problem relies on the randomness in the ideal world

Squared-Ratio Method: The Idea



attacker $\mathcal{A}$

- $\mathcal{A}$ is allowed to make q_m queries to each user $i \in [u]$
- Transcripts from the other users cannot contribute an information-theoretic adversary's query choice
  $\rightarrow$ the systems are mutually independent:

$$\mathrm{p}_{\mathcal{S}_i}(\mathbf{z}) = \prod_{j=1}^{u} \mathrm{p}_{\mathcal{S}_{i,j}}(z_j)$$

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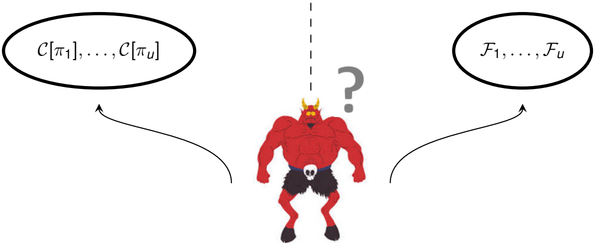

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Squared-Ratio Method [CCL23]

- For any good transcript z, it holds

$$\left| \frac{P_{S_1,1}(z)}{P_{S_0,1}(z)} - 1 \right| \leq \epsilon(z)$$

- Then we have

$$\|\mathrm{p}_{\mathcal{S}_1}(\cdot) - \mathrm{p}_{\mathcal{S}_0}(\cdot)\| \leq \sqrt{2u \cdot \mathbf{Ex}\left[\epsilon(z)^2\right]} + 2u \cdot \Pr[Z_{\mathcal{S}_0} \in \mathcal{T}_{\mathrm{bad}}]$$

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System of Equations — from Single Permutation

- A set of unknowns $\mathcal{P} = \{P_1, \ldots, P_{q_P}\}$ and knowns values $\lambda_1, \ldots, \lambda_q$
- A system of equations

$$\Gamma : \begin{cases} P_{\varphi(1)} \oplus P_{\varphi'(1)} = \lambda_1, \\ P_{\varphi(2)} \oplus P_{\varphi'(2)} = \lambda_2, \\ \qquad \vdots \\ P_{\varphi(q)} \oplus P_{\varphi'(q)} = \lambda_q, \end{cases}$$

where φ and φ' are two surjective index mappings such that

$$\varphi \colon \{1, \ldots, q\} \to \{1, \ldots, q_P\},$$
$$\varphi' \colon \{1, \ldots, q\} \to \{1, \ldots, q_P\},$$

- Mirror theory gives a lower bound on the number of solutions of these systems

Patarin's Mirror Theory

- Represents the system of equations by a graph
  - A distinct unknown $\rightarrow$ a vertex with unknown value
  - An equation $\rightarrow$ a λ-labeled edge

- Transcript graph should be
  - acyclic
  - non-zero path label (non-degenerate)

- In the "fine-tuned" ideal world, we need an additional condition: $\lambda \neq 0^n$

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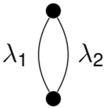

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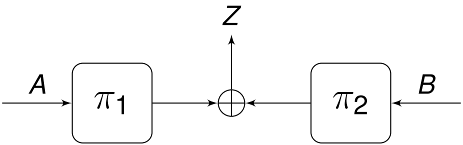
Framework For Use



- Query transcript $\tau = \{(A_1, B_1, Z_1), \ldots, (A_q, B_q, Z_q)\}$
- Each such algorithm consists of an evaluation of π_1 and an evaluation of π_2

$$\Gamma = \begin{cases} \pi_1(A_1) \oplus \pi_2(B_1) = Z_1, \\ \vdots \\ \pi_1(A_q) \oplus \pi_2(B_q) = Z_q. \end{cases}$$

- Define $\mathcal{T}_{\text{bad}}$ such that the graph is consistent
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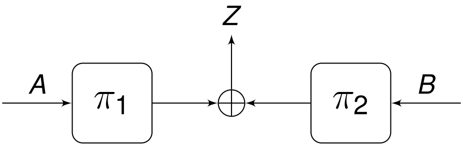

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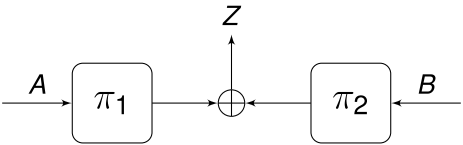

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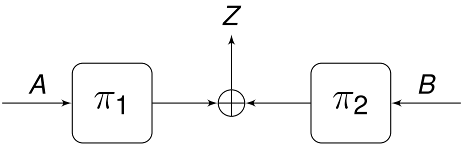

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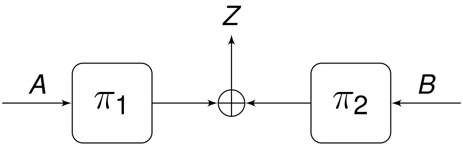

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