`Horst and Amaryllises`: Possible Generalizations of the Feistel and of the Lai-Massey Schemes

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CASA
Cyber Security in the Age of Large-Scale Adversaries

Ruhr Universität Bochum RUB
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Horst Görtz Institut

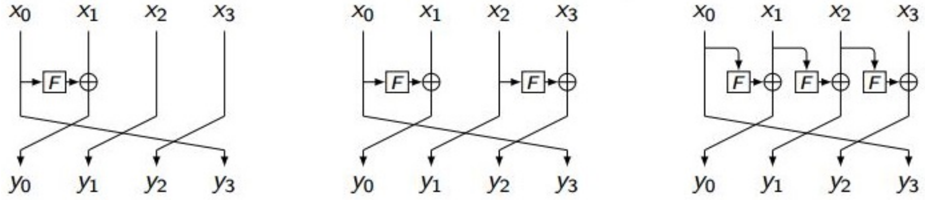
SPN and Feistel Schemes

The majority of the symmetric schemes follows one of the two following design strategies:

- Substitution Permutation Network (SPN):

$$(x_0, x_1, \ldots, x_{n-1}) \mapsto c + M \times (\mathsf{S}(x_0), \mathsf{S}(x_1), \ldots, \mathsf{S}(x_{n-1}))$$

- Feistel schemes:



Motivation

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New applications such as Multi-Party Computation (MPC), Fully Homomorphic Encryption (FHE), and Zero-Knowledge proofs (ZK) require symmetric-key primitives that

- are naturally defined over $\mathbb{F}_p^n$ for a large prime p (such as $p \approx 2^{128}$ or 2^{256})
- minimize their *multiplicative complexity*, i.e., the number of multiplications (= non-linear operations) required to compute and/or verify them.

Questions:

1. Is it possible to set up *new invertible non-linear layers* over $\mathbb{F}_q^n$ for $q = p^s$?
2. Is it possible to achieve better security argument and/or performances via them?

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From Feistel to Horst Schemes

Feistel scheme $\mathcal{F}$ over $\mathbb{F}_q^2$ (where $q = p^s$ for $s \geq 1$ and a prime $p \geq 2$):

$$(x_0, x_1) \mapsto (y_0, y_1) = (x_1, x_0 + \mathrm{F}(x_1)),$$

where $\mathrm{F} : \mathbb{F}_q \to \mathbb{F}_q$. *Always invertible independently of* F*:* $x_0 = y_1 - \mathrm{F}(x_1) = y_1 - \mathrm{F}(y_0)$.

Horst as a possible generalization:

$$(x_0, x_1) \mapsto (y_0, y_1) := (x_1, x_0 \cdot \mathrm{G}(x_1) + \mathrm{F}(x_1)),$$

where $\mathrm{G} : \mathbb{F}_q \to \mathbb{F}_q \setminus \{0\}$. Invertible as well:

$$(x_0, x_1) = \left(\frac{y_1 - \mathrm{F}(y_0)}{\mathrm{G}(y_0)}, y_0\right).$$

From Feistel to Horst Schemes

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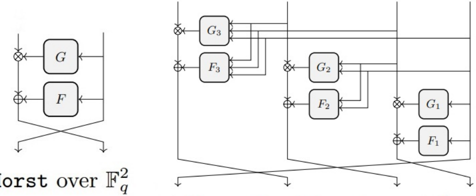
Horst Schemes over $\mathbb{F}_q^n$ [GHR+23]



Horst over $\mathbb{F}_q^2$

Generalized Horst over $\mathbb{F}_q^4$

The Horst scheme over $\mathbb{F}_q^n$:

$$y_i := \begin{cases} x_{i+1} \cdot \mathsf{G}^{(i+1)}(x_i, x_{i-1}, \ldots, x_0) + \mathsf{F}^{(i+1)}(x_i, x_{i-1}, \ldots, x_0) & \text{otherwise } (i \in \{0, 1, \ldots, n-2\}), \\ x_0 & \text{if } i = n-1, \end{cases}$$

where $\mathsf{F}^{(j)} : \mathbb{F}_q^j \to \mathbb{F}_q$ and $\mathsf{G}^{(j)} : \mathbb{F}_q^j \to \mathbb{F}_q \setminus \{\mathbf{0}\}$.

Constructing $G : \mathbb{F}_q \to \mathbb{F}_q \setminus \{0\}$ (1/3)

If q small (e.g., $q \approx 2^8$) and no particular condition on G: easy task! (E.g., brute force).
Our Goal: *construct low-degree* G *over any* $\mathbb{F}_q$.

1st Example: Let $q = p \geq 3$ be a prime. Remember: $x \mapsto x^2$ is not invertible.
Define

$$G(x) := x^2 + \alpha \cdot x + \beta$$

such that $\alpha, \beta \in \mathbb{F}_p$ satisfies $\alpha^2 - 4 \cdot \beta \neq z^2$ for each $z \in \mathbb{F}_p$.
Then, $G(x) = 0$ if and only if

$$x = (-\alpha \pm \sqrt{\alpha^2 - 4 \cdot \beta})/2\,,$$

which do not exist!

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Constructing $\mathsf{G} : \mathbb{F}_q \to \mathbb{F}_q \setminus \{0\}$ (2/3)

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Given a generic G over $\mathbb{F}_q$: very expensive to check if G returns zero or not!

Lemma 1

Let $\mathsf{G} : \mathbb{F}_q \to \mathbb{F}_q$ be such that (i) $\mathsf{G}(0) \neq 0$ and (ii) $\mathsf{H}(x) := x \cdot \mathsf{G}(x)$ is a permutation. Then, $\mathsf{G}(x) \neq 0$ for each $x \in \mathbb{F}_q$.

Proof.

Obviously, $\mathsf{H}(0) = 0$. Since H is a permutation, then $\mathsf{H}(x) \neq 0$ for each $x \in \mathbb{F}_q \setminus \{0\}$. Hence:

$$\forall x \in \mathbb{F}_q \setminus \{0\} : \qquad \mathsf{G}(x) = \frac{\mathsf{H}(x)}{x} \neq 0\,.$$

By assumption, $\mathsf{G}(0) \neq 0$. Hence, $\mathsf{G}(x) \neq 0$ for each x. □

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Constructing $G : \mathbb{F}_q \to \mathbb{F}_q \setminus \{0\}$ (3/3)

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2nd Example: Let $x \mapsto x^d$ be a permutation over $\mathbb{F}_q$, and let $\alpha \in \mathbb{F}_q \setminus \{0\}$. The function

$$G(x) := \frac{(x+\alpha)^d - \alpha^d}{x} = \sum_{i=1}^{d} \binom{d}{i} \cdot \alpha^i \cdot x^{d-1-i}$$

never returns zero (since $G(0) = d \cdot \alpha \neq 0$ and $G(x) \cdot x = (x+\alpha)^d - \alpha^d$ is invertible).

3rd Example: Over $\mathbb{F}_{2^n}$, let

$$G(x) = \sum_{i=0}^{d} \alpha_i \cdot x^{2^i - 1}.$$

for $\alpha_0 \neq 0$. Choose $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{F}_{2^n}$ such that $G(x) \cdot x = \sum_{i=0}^{d} \alpha_i \cdot x^{2^i}$ is invertible (note that $x \cdot G(x)$ is a linear map $\to$ easy to check!). Then, G never returns zero.

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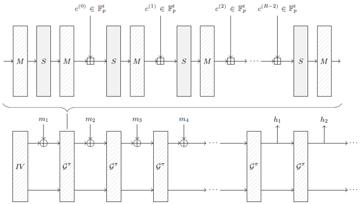
Feistel versus Horst in GRIFFIN [GHR+23] (1/2)



Non-linear layer $S(x_0, x_1, \ldots, x_{t-1}) = y_0 \| y_1 \| \ldots \| y_{t-1}$ over $\mathbb{F}_p^t$ defined as

$$y_i = \begin{cases} {x_0}^{1/d} & \text{if } i = 0, \\ {x_1}^{d} & \text{if } i = 1, \\ x_2 \cdot \left((L_i(y_0, y_1, 0))^2 + \alpha_2 \cdot L_i(y_0, y_1, 0) + \beta_2 \right) & \text{if } i = 2, \\ x_i \cdot \left((L_i(y_0, y_1, x_{i-1}))^2 + \alpha_i \cdot L_i(y_0, y_1, x_{i-1}) + \beta_i \right) & \text{otherwise,} \end{cases}$$

where $\gcd(d, p-1) = 1$ and $L_i : \mathbb{F}_p^3 \to \mathbb{F}_p$ are linear functions.

Feistel versus Horst in Griffin [GHR+23] (2/2)

$x \mapsto x^{1/d}$ is quite expensive $\longrightarrow$ *our goal:* minimize the number of rounds!

Griffin$^\times$ instantiated with Horst allows *stronger security argument* against Gröbner basis attacks w.r.t. Griffin$^+$ instantiated with Feistel:

- roughly speaking, cost of Gröbner basis attacks given by

$$\mathcal{O}\left(\binom{D_{\text{reg}} + n_v}{n_v}^2\right),$$

  where D_{reg} = degree of regularity, and n_v = number of variables;
- Griffin$^+$: D_{reg} remains almost constant w.r.t. the number of rounds $\longrightarrow$ *hard to estimate* the minimum number for guaranteeing security!
- Griffin$^\times$: D_{reg} *growths with the number of rounds.*

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Let $q = p^s$ be as before. Lai-Massey scheme $\mathcal{LM}$ over $\mathbb{F}_q^2$:

$$(x_0, x_1) \mapsto (y_0, y_1) = (\alpha \cdot (x_0 + \mathrm{F}(x_0 - x_1)), x_1 + \mathrm{F}(x_0 - x_1)) \; ,$$

where $\mathrm{F} : \mathbb{F}_q \to \mathbb{F}_q$ and $\alpha \neq 0$. $\mathcal{LM}$ is invertible independently of the details of F:

$$(x_0, x_1) = (y_0' - \mathrm{F}(y_0' - y_1), y_1 - \mathrm{F}(y_0' - y_1)) \, ,$$

since $x_0 - x_1 = y_0' - y_1$ where $y_0' = y_0/\alpha$.

Remark: $\alpha \notin \{0, 1\}$ crucial for *destroying the invariant subspace* $\langle [1, 1] \rangle \subseteq \mathbb{F}_q^2$.
($\alpha = 1$ fixed in the following for simplicity!)

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Let $n \geq 3$. For each $i \in \{0, 1, \ldots, n-2\}$, let $\lambda_0^{(i)}, \lambda_1^{(i)}, \ldots, \lambda_{n-1}^{(i)} \in \mathbb{F}_q$ be such that

$$\sum_{j=0}^{n-1} \lambda_j^{(i)} = 0 \,.$$

The Lai-Massey scheme over $\mathbb{F}_q^n$ is defined as $(x_0, x_1, \ldots, x_{n-1}) \mapsto (y_0, y_1, \ldots, y_{n-1})$ where

$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i := x_i + \mathrm{F}\left(\sum_{j=0}^{n-1} \lambda_j^{(0)} \cdot x_j, \sum_{j=0}^{n-1} \lambda_j^{(1)} \cdot x_j, \ldots, \sum_{j=0}^{n-1} \lambda_j^{(n-2)} \cdot x_j\right)$$

where $\mathrm{F} : \mathbb{F}_q^{n-1} \to \mathbb{F}_q$. As before, invertibility follows from $\sum_{j=0}^{n-1} \lambda_j^{(i)} \cdot x_j = \sum_{j=0}^{n-1} \lambda_j^{(i)} \cdot y_j$.

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Relation between Feistel and Lai-Massey Schemes

Theorem 2 ([Gra22])

A Lai-Massey scheme $\mathcal{LM}$ over $\mathbb{F}_q^n$ is extended Affine-Equivalent (AE) to a generalized Feistel scheme $\mathcal{F}$ over $\mathbb{F}_q^n$, that is, there exists two affine permutation A, B and an affine function C over $\mathbb{F}_q^n$ such that

$$\forall x \in \mathbb{F}_q^n : \qquad \mathcal{F}(x) = B \circ \mathcal{LM} \circ A(x) + C(x)\,.$$

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Proof of the Relation $\mathbb{F}_q^2$

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$\mathcal{LM}$ over $\mathbb{F}_q^2$ defined as $(x_0, x_1) \mapsto (x_0 + \mathsf{F}(x_0 - x_1), x_1 + \mathsf{F}(x_0 - x_1))$ is *affine-equivalent* to $(x_0, x_1) \mapsto (x_1 + \mathsf{F}(x_0), x_0)$ via the invertible linear transformations

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and $C = 0$. Indeed:

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \xrightarrow{A \times \cdot} \begin{bmatrix} x_0 - x_1 \\ x_1 \end{bmatrix} \xrightarrow{\mathcal{F}(\cdot)} \begin{bmatrix} x_1 + \mathsf{F}(x_0 - x_1) \\ x_0 - x_1 \end{bmatrix} \xrightarrow{B \times \cdot} \begin{bmatrix} x_0 + \mathsf{F}(x_0 - x_1) \\ x_1 + \mathsf{F}(x_0 - x_1) \end{bmatrix} .$$

(See [Gra22] for the proof regarding the general case $\mathbb{F}_q^n$.)

Open Questions

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- Is it possible to transfer the results published in the literature (including indistinguishability, indifferentiability, ...) for Feistel schemes to Lai-Massey schemes?
- Is it possible to *generalize* the Lai-Massey scheme such that the (extended) affine equivalence to Feistel schemes does **not** hold?

Generalized Lai-Massey Scheme

Definition 3 ([Gra22])

Let $n \geq 2$. For each $i \in \{0, 1, \ldots, n-2\}$, let $\lambda_0^{(i)}, \lambda_1^{(i)}, \ldots, \lambda_{n-1}^{(i)} \in \mathbb{F}_q$ be s.t. $\sum_{l=0}^{t-1} \lambda_l^{(i)} = 0$.

We say that the scheme $(x_0, x_1, \ldots, x_{n-1}) \mapsto (y_0, y_1, \ldots, y_{n-1}) := \mathcal{LM}^G(x_0, x_1, \ldots, x_{n-1})$ where

$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i := x_i + \mathsf{F}_i(z_0, z_1, \ldots, z_{n-2})$$

such that $\mathsf{F}_0, \mathsf{F}_1, \ldots, \mathsf{F}_{n-1} : \mathbb{F}_q^{n-1} \to \mathbb{F}_q$ and

$$\forall j \in \{0, 1, \ldots, n-2\} : \qquad z_j := \sum_{l=0}^{n-1} \lambda_l^{(j)} x_l$$

is a “Generalized Lai-Massey” scheme **if** it is invertible.

Generalized Lai-Massey Scheme: Example

Let $p \geq 3$. The generalized Lai-Massey scheme defined as

$$\begin{aligned} y_0 &= x_0 + F^{(1)}(x_0 - x_1) + F^{(3)}(x_0 - x_1, x_1 - x_2, x_2 - x_3) \\ y_1 &= x_1 + F^{(1)}(x_0 - x_1) + F^{(3)}(x_0 - x_1, x_1 - x_2, x_2 - x_3) \\ y_2 &= x_2 + F^{(2)}(x_0 - x_1, x_2 - x_3) + F^{(3)}(x_0 - x_1, x_1 - x_2, x_2 - x_3) \\ y_3 &= x_3 + F^{(2)}(x_0 - x_1, x_2 - x_3) + F^{(3)}(x_0 - x_1, x_1 - x_2, x_2 - x_3) \end{aligned}$$

for $F^{(i)} : \mathbb{F}_p^i \to \mathbb{F}_p$ with $i \in \{1, 2, 3\}$ is

- invertible:

$$x_0 - x_1 = y_0 - y_1 ; \qquad x_2 - x_3 = y_2 - y_3 ;$$
$$x_1 - x_2 = y_1 - y_2 - F^{(1)}(y_0 - y_1) - F^{(2)}(y_0 - y_1, y_2 - y_3) ;$$

- **not** extended affine equivalent to any Feistel scheme (see [Gra22] for the proof).

Redundant Lai-Massey Scheme

Definition 4 ([Gra22])

Let $n \geq 2$.
We say that the scheme $(x_0, x_1, \ldots, x_{n-1}) \mapsto (y_0, y_1, \ldots, y_{n-1}) := \mathcal{LM}^R(x_0, x_1, \ldots, x_{n-1})$ where

$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i := x_i + \mathsf{F}(x_0, x_1, \ldots, x_{n-1})$$

such that $\mathsf{F} : \mathbb{F}_q^n \to \mathbb{F}_q$ is a "Redundant Lai-Massey" scheme **if** it is invertible.

Redundant Lai–Massey Scheme: Example (1/2)

Let $p \geq 3$ be a prime integer. Let $\alpha \in \mathbb{F}_p \setminus \{0\}$ be such that $-2 \cdot \alpha \neq z^2$ for each $z \in \mathbb{F}_p$. Then,

$$(x_0, x_1) \mapsto (y_0, y_1) := (x_0 + z, x_1 + z)$$

where

$$z := \alpha \cdot (x_0 - x_1)^2 \cdot (x_0 + x_1)$$

is a redundant Lai-Massey scheme. Invertibility follows from:

- $y_0 - y_1 = x_0 - x_1$;
- hence, $y_0 + y_1 = (x_0 + x_1) \cdot (1 + 2\alpha \cdot (y_0 - y_1)^2)$, which implies

$$x_0 + x_1 = \frac{y_0 + y_1}{1 + 2\alpha \cdot (y_0 - y_1)^2}$$

where $1 + 2\alpha \cdot z^2 \neq 0$ by assumption on α.

Redundant Lai–Massey Scheme: Example (1/2)

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Redundant Lai-Massey Scheme: Example (2/2)

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It is **not** extended affine equivalent to any Feistel scheme over $\mathbb{F}_p^2$.

Roughly speaking, reason:

- Feistel $\mathcal{F}(x_0, x_1) = (x_1 + \mathrm{F}(x_0), x_0)$: F depends on one input only;
- redundant Lai-Massey $\mathcal{LM}^R(x_0, x_1) = (x_0 + \mathrm{G}(x_0, x_1), x_1 + \mathrm{G}(x_0, x_1))$ just proposed:

$$\mathrm{G}(x_0, x_1) = \alpha \cdot (x_0 - x_1)^2 \cdot (x_0 + x_1)$$

  depends on two inputs.

(Note that $x_0 - x_1$ and $x_0 + x_1$ are linearly independent)

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Amaryllises Schemes [Gra22]

Let $q = p^s$ as before, and let $n \geq 2$. Let

1. $\mathrm{F} : \mathbb{F}_q \to \mathbb{F}_q$ be a function s.t. (i) $\mathrm{F}(0) \neq 0$ and (ii) $\mathrm{G}(x) := x \cdot \mathrm{F}(x)$ is invertible over $\mathbb{F}_q$ *(similar to before!)*;
2. $\mathrm{H} : \mathbb{F}_q^{n-1} \to \mathbb{F}_q$ be any function;
3. $\beta_0, \beta_1, \ldots, \beta_{n-1} \in \mathbb{F}_q \setminus \{0\}$ s.t. $\sum_{i=0}^{n-1} \beta_i = 0$ **if** H is *not* identically equal to zero ;
4. $\forall j \in \{0, 1, \ldots, n-2\}$, let $\{\gamma_i^{(j)}\}_{i \in \{0,1,\ldots,n-1\}}$ be s.t. $\gamma_i^{(j)} \in \mathbb{F}_q$ and $\sum_{i=0}^{n-1} \gamma_i^{(j)} = 0$.

The Amaryllises scheme $\mathcal{A}$ over $\mathbb{F}_q^n$ defined as $\mathcal{A}(x_0, x_1, \ldots, x_{t-1}) := y_0 \| y_1 \| \ldots \| y_{n-1}$ where

$$y_i = x_i \cdot \mathrm{F}\left(\sum_{j=0}^{n-1} \beta_j \cdot x_j\right) + \mathrm{H}\left(\sum_{j=0}^{n-1} \gamma_j^{(0)} \cdot x_j, \ldots, \sum_{j=0}^{n-1} \gamma_j^{(n-2)} \cdot x_j\right)$$

for each $i \in \{0, 1, \ldots, n-1\}$ is invertible.

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Invertibility of Amaryllises Schemes

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Invertibility follows from:

- recover $\sum_{i=0}^{n-1} \beta_i \cdot x_i$ by exploiting (1) and (3):

$$\sum_{i=0}^{n-1} \beta_i \cdot y_i = \left(\sum_{i=0}^{n-1} \beta_i \cdot x_i\right) \cdot \mathrm{F}\left(\sum_{i=0}^{n-1} \beta_i \cdot x_i\right) + \mathrm{H}\left(\sum_{i=0}^{n-1} \gamma_i^{(0)} \cdot x_i, \ldots, \sum_{i=0}^{n-1} \gamma_i^{(n-1)} \cdot x_i\right) \cdot \underbrace{\sum_{i=0}^{n-1} \beta_i}_{=0}$$

$$= \mathrm{G}\left(\sum_{i=0}^{n-1} \beta_i \cdot x_i\right) \quad \longrightarrow \quad \sum_{i=0}^{n-1} \beta_i \cdot x_i = \mathrm{G}^{-1}\left(\sum_{i=0}^{n-1} \beta_i \cdot y_i\right);$$

- given $\sum_{i=0}^{n-1} \beta_i \cdot x_i$ and since $\mathrm{F}(z) \neq 0$ for each $z \in \mathbb{F}_q$, recover $\sum_{j=0}^{n-1} \gamma_j^{(i)} \cdot x_j$ as in a standard Lai-Massey scheme.

Contracting-Amaryllises Schemes [Gra22]

Let $q = p^s$ as before, and let $n \geq 2$. Let

1. $e \geq 1$ be an integer such that $\gcd(e, q-1) = 1$;
2. $F : \mathbb{F}_q^n \to \mathbb{F}_q \setminus \{0\}$ be a function that (i) never returns zero for any non-zero input (i.e., $F(x) = 0$ if only if $x = 0 \in \mathbb{F}_q^n$), and s.t. (ii) the function $G_{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}}(x) : \mathbb{F}_q \to \mathbb{F}_q$ defined as
$$G_{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}}(x) := x^e \cdot F(\alpha_0 \cdot x, \alpha_1 \cdot x, \ldots, \alpha_{n-1} \cdot x)$$
is invertible for each arbitrary fixed non-null $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{F}_q^n \setminus \{(0, 0, \ldots, 0)\}$.

The Contracting-Amaryllises scheme $\mathcal{A}_C$ over $\mathbb{F}_q^n$ defined as $\mathcal{A}_C(x_0, x_1, \ldots, x_{n-1}) = y_0 \| y_1 \| \ldots \| y_{n-1}$ where

$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i = x_i^e \cdot F(x_0, x_1, \ldots, x_{n-1})$$

is invertible.

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The Contracting-Amaryllises scheme $\mathcal{A}_C$ over $\mathbb{F}_q^n$ defined as $\mathcal{A}_C(x_0, x_1, \ldots, x_{n-1}) = y_0 \| y_1 \| \ldots \| y_{n-1}$ where

$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i = x_i^e \cdot F(x_0, x_1, \ldots, x_{n-1})$$

is invertible.

Invertibility of Contracting-Amaryllises Schemes

Note:

1. $y_i = 0 \qquad \longleftrightarrow \qquad x_i = 0$;
2. since F never returns zero, then for each $i, j \in \{0, 1, \ldots, n-1\}$:

$$y_i \cdot x_j^e = y_j \cdot x_i^e \,.$$

Assume $y_i \neq 0$. Since $x \mapsto x^e$ is invertible, then

$$\begin{aligned} y_i &= x_i^e \cdot \mathrm{F}\left(\left(\frac{y_0}{y_i}\right)^{\frac{1}{e}} \cdot x_i, \ldots, \left(\frac{y_{i-1}}{y_i}\right)^{\frac{1}{e}} \cdot x_i, x_i, \left(\frac{y_{i+1}}{y_i}\right)^{\frac{1}{e}} \cdot x_i, \ldots, \left(\frac{y_{n-1}}{y_i}\right)^{\frac{1}{e}} \cdot x_i\right) \\ &\equiv \mathrm{G}_{\left(\frac{y_0}{y_i}\right)^{\frac{1}{e}}, \ldots, \left(\frac{y_{i-1}}{y_i}\right)^{\frac{1}{e}}, 1, \left(\frac{y_{i+1}}{y_i}\right)^{\frac{1}{e}}, \ldots, \left(\frac{y_{n-1}}{y_i}\right)^{\frac{1}{e}}}(x_i) \,. \end{aligned}$$

By inverting G, it is possible to find x_i.

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Constructing $G_{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}}(x) : \mathbb{F}_q \to \mathbb{F}_q$

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Let $d \geq 3$ be such that $\gcd(d, q-1) = 1$. Let $F : \mathbb{F}_q^n \to \mathbb{F}_q$ be an **homogeneous function** of degree $d - e$ (i.e., it contains only monomial of degree $d - e$) such that $F(x) = 0$ if only if $x = 0 \in \mathbb{F}_q^n$. Then:

$$\begin{aligned} G_{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}}(x) &= x^e \cdot F(\alpha_0 \cdot x, \alpha_1 \cdot x, \ldots, \alpha_{n-1} \cdot x) \\ &= x^d \cdot F(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \end{aligned}$$

is invertible (since $x \mapsto x^d$ is invertible, and $F(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \neq 0$ by assumption).

(See [Gra22] for other concrete examples.)

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Summary and Open Problems

New invertible non-linear layers for symmetric schemes!

Open Problems:

1. exploit the EA-equivalence (and/or the CCZ one?) between Feistel and Lai-Massey schemes for transferring known results;
2. analyze the *statistical and the algebraic cryptographic properties* of the proposed schemes:
   - an initial study proposed in [Gra22] and in [RS22];
   - what about the degree of regularity over multiple rounds?
3. analyze the *impact* on the design: is it possible to improve the performances of current schemes?
4. set up other invertible non-linear schemes.

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Thanks for your attention!

Questions?

Comments?